An understanding of differentially rotating relativistic stars is key to many areas of astrophysics, in particular to the emission of gravitational waves. A newly born, proto-neutron star or a compact remnant of neutron stars binary merger are expected to rotate differentially and to be important sources of gravitational radiation. A highly accurate and stable, relativistic, multidomain spectral code is used to explore the whole solution space for broad ranges of the degree of differential rotation. Staying within an astrophysically motivated range of rotation profiles, we investigate the characteristics of neutron stars with maximal mass for all types of families of differentially rotating neutron stars and different equations of state.We find various types of configurations, which were not considered in previous work, mainly due to numerical limitations. The maximum allowed mass for the new types of configurations and moderate degree of differential rotation can be even 2-4 times higher then the maximum mass of non-rotating neutron stars with the same equation of state. Differential rotation can temporarily stabilize a hyper-massive neutron star against gravitational collapse. This result may have important consequences for the gravitational wave signal expected from coalescing neutron star binaries or from some supernova events.